

The 12-th roots of the discriminant of an elliptic curve and the torsion points

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Abstract

Given an elliptic curve over a field of characteristic different from 2,3, its discriminant defines a μ_{12} -torsor over the field. In this paper, we give an explicit description of this μ_{12} -torsor in terms of the 3-torsion points and of the 4-torsion points on the given elliptic curve. As an application, we generalize a result of Coates on the 12-th root of the discriminant of an elliptic curve.

1 Introduction

Let E be an elliptic curve over a field K . Then, the *discriminant* Δ_E of E is defined to be a value in $K^\times/(K^\times)^{12}$, which depends only on the isomorphism class of E .

To study Δ_E , it suffices to consider $\Delta_E \bmod (K^\times)^n$ for $n = 3, 4$ separately, because $K^\times/(K^\times)^{12} \simeq K^\times/(K^\times)^3 \times K^\times/(K^\times)^4$. In the rest of this introduction, we suppose that $n = 3, 4$ and that K be a field of characteristic prime to n . Then, we may focus on the μ_n -torsor $\mu_n \sqrt[n]{\Delta_E}$ over K consisting of n -th roots of Δ_E since $\Delta_E \bmod (K^\times)^n$ corresponds to the μ_n -torsor $\mu_n \sqrt[n]{\Delta_E}$ via the isomorphism $K^\times/(K^\times)^n \cong H^1(K, \mu_n)$ from Kummer theory.

The goal of this paper is to give an explicit description of the μ_n -torsor $\mu_n \sqrt[n]{\Delta_E}$ out of $E[n]$. More precisely, we construct a $\bigwedge^2 E[n]$ -torsor $T_n(E[n])$ over K and define a canonical isomorphism w_n from $T_n(E[n])$ to the μ_n -torsor $\mu_n \sqrt[n]{\Delta_E}$ over K such that the following diagram is commutative:

$$\begin{array}{ccc} \bigwedge^2 E[n] \times T_n(E[n]) & \xrightarrow{\text{action}} & T_n(E[n]) \\ e_n \times w_n \downarrow & & \downarrow w_n \\ \mu_n \times \mu_n \sqrt[n]{\Delta_E} & \xrightarrow{\text{action}} & \mu_n \sqrt[n]{\Delta_E}, \end{array}$$

where e_n is the Weil pairing normalized as in Remark 2.8 and Remark 2.10.

We now describe the organization of this paper.

In Section 2, we give a brief review of several results on elliptic curves. The results on Tate curves and modular curves will be crucial in the proof of our main theorem.

Sections 3 and 4 give a preparation for stating our main theorem. Our first task is to construct a $\bigwedge^2 E[n]$ -torsor $T_n(E[n])$, which are explained in Section 3 as we mentioned above. Since this construction only uses the fact $E[n] \cong (\mathbb{Z}/(n))^2$, Section 3 deals with an abstract free $\mathbb{Z}/(n)$ -module of rank 2. The second task is to construct a bijection $w_n : T_n(E[n]) \rightarrow \mu_n \sqrt[n]{\Delta_E}$, which is done

in Section 4. Also, we give a simple description of the action of $\bigwedge^2 E[n]$ on $T_n(E[n])$ when E is a Tate curve. The constructions of w_n are based on [4] and [7].

Next, integrating the results of the previous two sections, we proceed to the main part of this paper: in Section 5, we give the precise statement of the main theorem and prove it. In its proof, using the modular curves of level 3 and 4, we reduce a claim on general elliptic curves to a claim on a single Tate curve. As such an elliptic curve, we choose the Tate curve over a Laurent series field $\mathbb{Q}((q))$. This choice enables us to compute concretely both the map $w_n : T_n(E[n]) \rightarrow \mu_n \sqrt[n]{\Delta_E}$ and the Weil pairing for the Tate curve.

In the last section, we give a consequence of our main theorem extending the result of Coates [2] in the following sense:

Corollary 6.1. *Let E and E' be elliptic curves over K of characteristic $\text{char}(K) \neq 2, 3$, and $\varphi : E \rightarrow E'$ be an isogeny over K . If $d = \deg \varphi$ is prime to 12, then we have $\Delta_E = (\Delta_{E'})^d$ in $K^\times / (K^\times)^{12}$.*

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Convention and Terminology

Let $\varphi : G \rightarrow H$ be a homomorphism of groups and $f : X \rightarrow Y$ be a map, where X (resp. Y) is a G -set (resp. H -set). We say that f is *compatible with φ* if for any $x \in X$ and $g \in G$ we have $f(g \cdot x) = \varphi(g) \cdot f(x)$.

Also, we often denote by ω (resp. i) a primitive cubic (resp. 4th) root of unity.

2 Review on elliptic curves

In this section, we review some results on elliptic curves we need in this paper. We omit all the proofs of these results, and only give references.

2.1 Elliptic curves

Definition 2.1. *Let S be a scheme. An elliptic curve over S is a pair (E, O) such that E is a proper smooth scheme over S whose geometric fibers are connected algebraic curves of genus 1, and that O is a section of $E \rightarrow S$.*

If there is no risk of confusion, we omit to write O and simply call E an elliptic curve. It is well-known as the Abel-Jacobi theorem (THEOREM 2.1.2, [3]) that every elliptic curve (E, O) admits a structure of a commutative group scheme with O the unit section. For each positive integer N , we denote by $E[N]$ the N -torsion subgroup scheme of E .

Remark 2.2. If N is invertible on S , then $E[N]$ is a finite étale commutative group scheme. If, in addition, S is a spectrum of a field K , then we often identify the finite étale commutative group scheme $E[N]$ over K with the associated $G_K = \text{Gal}(\bar{K}/K)$ -module $E[N](\bar{K})$.

Definition 2.3. For a Weierstrass equation over a ring A of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

we define $b_i \in A$ ($i = 2, 4, 6, 8$) by

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, \\ b_4 &= a_3^2 + 4a_6, \\ b_6 &= 2a_4 + a_1a_3, \\ b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 + a_4^2, \end{aligned}$$

and define the discriminant Δ of the Weierstrass equation by

$$\Delta := 9b_2b_4b_6 - b_2^2b_8 - 8b_4^3 - 27b_6^2.$$

.

The following theorem actually characterizes the definition of elliptic curves.

Theorem 2.4. Let E/S be an elliptic curve. Then there exists an affine covering $\{U_i\}_{i \in I}$ of S such that, for each $i \in I$, E_{U_i}/U_i is given by a Weierstrass equation over $\Gamma(U_i, \mathcal{O})$ with the discriminant invertible on $\Gamma(U_i, \mathcal{O})$.

Proof. See (2,2), [3] □

Remark 2.5. (1) It is well-known (for example, see [8]) that, for a Weierstrass equation, the only change of variables fixing $[0 : 1 : 0]$ and preserving the general Weierstrass form is

$$\begin{aligned} x &= u^2X + r, \\ y &= u^3Y + sX + t, \end{aligned} \tag{2.1}$$

with $u \in A^\times$ and $r, s, t \in A$. Let b_i and Δ (b'_i and Δ' , respectively) be the constants defined in Definition 2.1 for the Weierstrass equation with coordinates (x, y) ((X, Y) , respectively). Then the above change of variable gives the following formula:

$$\begin{aligned} u^2b'_2 &= b_2 + 12r, \\ u^4b'_4 &= b_4 + rb_2 + 6r^2, \\ u^{12}\Delta' &= \Delta. \end{aligned} \tag{2.2}$$

(2) Suppose that 2 is invertible on a ring A . Then, any Weierstrass equation can be made into the form

$$y^2 = x^3 + a_2x^2 + a_4x + a_6$$

over A , and

$$x = u^2X + r, \quad y = u^3Y \quad (u \in A^\times, r \in A)$$

is the only change of variables fixing the point $[0 : 1 : 0]$ and preserving such a Weierstrass form.

Let E be an elliptic curve over a field K . Take a Weierstrass equation for E and consider its discriminant Δ . The Remark 2.5 (1) implies that the image of Δ in $K^\times/(K^\times)^{12}$ only depends on E . We denote it by $\Delta_E \in K^\times/(K^\times)^{12}$, and call it *the discriminant of E* .

2.2 The modular curve of level $r \geq 3$

In the following, we recall the representability of a moduli problem on elliptic curves. This result will be used in the proof of our main theorem to reduce Proposition 5.1 to the Tate curve case (Lemma 4.12).

Theorem 2.6. *For a positive integer r , let $\mathcal{M}(r) : \text{Sch}/\mathbb{Z}[1/r] \rightarrow \text{Sets}$ be a functor defined as follows; for a scheme S over $\mathbb{Z}[1/r]$, $\mathcal{M}(r)(S)$ is the set of isomorphism classes of the pair (E, α) , where E is an elliptic curve over S and $\alpha : (\mathbb{Z}/(r))^2 \xrightarrow{\cong} E[r]$ is an isomorphism of group schemes over S .*

(1) *If $r = 1$, then the morphism $j : \mathcal{M} \rightarrow \mathbb{A}_{\mathbb{Z}}^1$ taking the j -invariant makes $\mathbb{A}_{\mathbb{Z}}^1$ into the coarse moduli of $\mathcal{M} := \mathcal{M}(1)$.*

(2) *If $r \geq 3$, then $\mathcal{M}(r)$ has the fine moduli $Y(r)$, which is a connected smooth curve over $\mathbb{Z}[1/r]$.*

Proof. See Corollary 8.40, [5] for (1) and Lemma 8.37, [5] for (2). \square

In this paper, we call $Y(r)$ *the modular curve of level r* .

2.3 Weil pairing

Next, we recall the Weil pairing of an elliptic curve. We make explicit the sign convention of the pairing, which is an important point in our result.

Theorem 2.7. *Let N be a positive integer. For any elliptic curve E/S , there exists a canonical bilinear pairing*

$$e_N : E[N] \times E[N] \longrightarrow \mu_N,$$

which is alternating and induces a self-duality of $E[N]$. The construction of this pairing is functorial in the sense that it defines a morphism of functors

$$\begin{aligned} \mathcal{M}(N) &\longrightarrow \mu_N. \\ [(E/S, (P, Q))] &\rightsquigarrow e_N(P, Q) \end{aligned}$$

Proof. See (2.8), [3]. \square

Remark 2.8. There are two choices of the sign of e_N . We choose e_N so that it satisfies the following equality: for any elliptic curve $E \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ ($\tau \in \mathcal{H}$) over \mathbb{C} , we have

$$e_N \left(\frac{1}{N}, \frac{\tau}{N} \right) = \exp \left(\frac{2\pi i}{N} \right).$$

We call the pairing of such a choice the normalized Weil pairing.

2.4 The Tate curve

Finally, we recall some properties of the Tate curves. The main references on this topic are [6] and [8].

Let (K, v) be a complete discrete valuation field. We fix an element $q \in K^\times$ satisfying $v(q) > 0$. Consider the elliptic curve (called the Tate curve) over K defined by the following Weierstrass equation:

$$E_q : y^2 + xy = x^3 + \left(-5 \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n} \right) x + \left(-\frac{1}{12} \sum_{n \geq 1} \frac{(7n^5 + 5n^3)q^n}{1 - q^n} \right).$$

The discriminant Δ of this Weierstrass equation is given by

$$\Delta = q \prod_{m \geq 1} (1 - q^m)^{24}. \quad (2.3)$$

The curve E_q has the following important properties:

Theorem 2.9. *Let the notation be as above. Let u be an indeterminate, and $x = x(u, q)$ and $y = y(u, q)$ be the two power series in $\mathbb{Z}[u, u^{-1}][[q]]$ defined by*

$$x(u, q) := \sum_{n \in \mathbb{Z}} \frac{q^n u}{(1 - q^n u)^2} - 2 \sum_{n \geq 1} \frac{n q^n}{1 - q^n} \quad (2.4)$$

$$y(u, q) := \sum_{n \in \mathbb{Z}} \frac{q^{2n} u^2}{(1 - q^n u)^3} + \sum_{n \geq 1} \frac{n q^n}{1 - q^n}. \quad (2.5)$$

(1) Then the map

$$\begin{aligned} K^\times / q^\mathbb{Z} &\longrightarrow E_q(K) \\ u &\longmapsto (x(u, q), y(u, q)) \end{aligned}$$

is an isomorphism of abelian groups.

More generally, the map

$$\begin{aligned} \bar{K}^\times / q^\mathbb{Z} &\longrightarrow E_q(\bar{K}) \\ u &\longmapsto (x(u, q), y(u, q)) \end{aligned}$$

makes sense and is an isomorphism of G_K -modules.

(2) For each positive integer N prime to the characteristic of K , there exists a canonical exact sequence

$$0 \rightarrow \mu_N(\bar{K}) \rightarrow E_q[N](\bar{K}) \rightarrow \mathbb{Z}/(N) \rightarrow 0 \quad (2.6)$$

of G_K -modules.

Remark 2.10. Let us consider here the Tate curve E_q over $K = \mathbb{Q}((q))$. Then, the exact sequence (2.5) does not split as G_K -modules. However, consider the base change of E_q/K by the injection

$$\begin{aligned} K &\longrightarrow L := \mathbb{Q}(\zeta_N)((z)) \\ q &\longmapsto z^N. \end{aligned}$$

Then, we have the isomorphism of G_L -modules

$$\begin{aligned}\mathbb{Z}/(N) &\xrightarrow{\cong} \mu_{N,L}(\bar{L}) \\ 1 &\mapsto \zeta_N.\end{aligned}$$

We also obtain the canonical section

$$\begin{aligned}\mathbb{Z}/(N) &\longrightarrow E_{q,L}[N](\bar{L}). \\ 1 &\mapsto z\end{aligned}$$

of the homomorphism $E_{q,L}[N](\bar{L}) \rightarrow \mathbb{Z}/(N)$ in the exact sequence (2.5). Therefore, the exact sequence splits and gives a canonical identification

$$\begin{aligned}E_{q,L}[N] &\cong \mu_{N,L} \times \mathbb{Z}/(N) \\ &\cong \mathbb{Z}/(N) \times \mathbb{Z}/(N).\end{aligned}$$

Under this identification, it is known that the normalized Weil pairing e_N sends each $((a, b), (c, d)) \in (\mathbb{Z}/(N) \times \mathbb{Z}/(N))^2$ to ζ_N^{ad-bc} ; that is, $e_N(\zeta_N, z) = \zeta_N$. (See [1, VII, 1, 16].)

3 Construction of torsors

Our description of the 12-th roots of the discriminant of an elliptic curve by the torsion points consists of the following two parts: one needs some geometric properties of elliptic curves, and the other only needs a bit knowledge of linear algebra. In this section, we explain the linear-algebraic part. Throughout this section, we assume that $n = 3, 4$, and let V denote a free $\mathbb{Z}/(n)$ -module of rank 2. Our tasks here are to construct a set $T_n(V)$, which will later describe $\mu_n \sqrt[n]{\Delta_E}$ when $V = E[n]$, and to define a simply transitive action of $\bigwedge^2 V$ on $T_n(V)$.

3.1 The construction of $T_3(V)$

Here, V is a 2-dimensional vector space over \mathbb{F}_3 . We construct a set $T_3(V)$ attached to V . We write $\mathbb{P}(V) := (V \setminus \{0\})/\{\pm 1\}$ for the projective line associated with V , and \overline{P} for the image of a point $P \in V \setminus \{0\}$ in $\mathbb{P}(V)$. Note that $\#\mathbb{P}(V) = 4$.

Definition 3.1. *We define a set $T_3(V)$ by*

$$T_3(V) = \{\{X, Y\} \mid X \sqcup Y = \mathbb{P}(V), \#X = \#Y = 2\}.$$

That is, this is the set of 2-2 partitions of $\mathbb{P}(V)$.

For a basis (P, Q) of V , we write $[P, Q] \in T_3(V)$ for $\{\{\overline{P}, \overline{Q}\}, \{\overline{P+Q}, \overline{P-Q}\}\}$. Since $\mathbb{P}(V) = \{\overline{P}, \overline{Q}, \overline{P+Q}, \overline{P-Q}\}$, the set $T_3(V)$ consists of 3 elements $[P, Q]$, $[P, P+Q]$, and $[P, P-Q]$.

Remark 3.2. For the later use, note that $T_3(V)$ has canonically a simply transitive action of the alternating group $\mathcal{A}(\text{Aut}(T_3(V)))$.

3.2 The construction of $T_4(V)$

In this subsection, we define a set $T_4(V)$ for a free $\mathbb{Z}/(4)$ -module V of rank 2. Our construction of $T_4(V)$ for $n = 4$ is motivated by the definition of w_4 in Subsection 4.2.

Denote $V[2] = \ker(2 \times : V \rightarrow V)$ and define a set $S_4(V)$ by

$$S_4(V) = \{(P, Q, R) \in V^3 \mid \{2P, 2Q, 2R\} = V[2] \setminus 0\}.$$

For each $\sigma \in \mathfrak{S}_3$, define an involution $[\sigma]$ of $S_4(V)$ by $[\sigma](P_1, P_2, P_3) := (P'_1, P'_2, P'_3)$ with

$$P'_{\sigma(1)} = P_{\sigma(1)}, P'_{\sigma(2)} = P_{\sigma(2)} + 2P_{\sigma(1)}, \text{ and } P'_{\sigma(3)} = P_{\sigma(3)}.$$

Since $[\sigma]$'s commute with each other, we obtain an action of $\mathbb{F}_2^{\mathfrak{S}_3}$ on $S_4(V)$. Combining this action with a canonical action of \mathfrak{S}_3 on $S_4(V)$ by permutations, we obtain an action of $G := \mathfrak{S}_3 \ltimes \mathbb{F}_2^{\mathfrak{S}_3}$ on $S_4(V)$. Here, the left action of \mathfrak{S}_3 on $\mathbb{F}_2^{\mathfrak{S}_3}$ defining the semidirect product is given by $\tau \cdot [\sigma] = [\tau\sigma]$ for $\sigma, \tau \in \mathfrak{S}_3$. Let N be the kernel of the composite

$$r : \mathbb{F}_2^{\mathfrak{S}_3} \rightarrow \mathbb{F}_2^{\mathfrak{A}_3} \rightarrow \{\pm 1\},$$

where the first map is the restriction to \mathfrak{A}_3 and the second map is the one sending $\sum a_\sigma [\sigma]$ to $(-1)^{\sum a_\sigma}$. Since the action of \mathfrak{S}_3 on $\mathbb{F}_2^{\mathfrak{S}_3}$ defining the semidirect product induces an action of \mathfrak{A}_3 on N , we obtain $H := \mathfrak{A}_3 \ltimes N \triangleleft G$.

Definition 3.3. Under the above setting, define the set $T_4(V)$ by the quotient

$$T_4(V) := H \setminus S_4(V).$$

The equivalence class of $(P, Q, R) \in S_4(V)$ is denoted by $[P, Q, R] \in T_4(V)$.

Note that $T_4(V)$ has an action of $G/H = \mathfrak{S}_3/\mathfrak{A}_3 \times \{\pm 1\}$.

Remark 3.4. (1) The element $-1 \in G/H$ acts on $T_4(V)$ by

$$[P, Q, R] \mapsto -[P, Q, R] := [-P, -Q, -R].$$

Indeed, $-1 = r(\sum_{\sigma \in \mathfrak{S}_3} [\sigma])$ and

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_3} [\sigma](P, Q, R) &= (P + 2Q + 2R, Q + 2R + 2P, R + 2P + 2Q) \\ &= (-P, -Q, -R) \end{aligned}$$

in $S_4(V)$.

(2) Define a set $S_2(V[2])$ by

$$S_2(V[2]) = \{(A, B, C) \in V[2]^3 \mid \{A, B, C\} = V[2] \setminus 0\}$$

equipped with the canonical action by \mathfrak{S}_3 . Consider the quotient

$$T_2(V[2]) := \mathfrak{A}_3 \setminus S_2(V)$$

by the cyclic permutations; that is, it is the set of the cyclic orders on $V[2] \setminus 0$. The equivalence class of an element $(A, B, C) \in S_2(V[2])$ is denoted by

$[A, B, C] \in T_2(V[2])$.

The map

$$\begin{aligned}\tilde{\pi} : S_4(V) &\longrightarrow S_2(V[2]) \\ (P, Q, R) &\mapsto (2P, 2Q, 2R)\end{aligned}$$

is compatible with $\text{pr}_1 : G \rightarrow \mathfrak{S}_3$. Hence, it induces a map

$$\pi : T_4(V) \longrightarrow T_2(V[2]),$$

which is compatible with $\text{pr}_1 : G/H \rightarrow \mathfrak{S}_3/\mathfrak{A}_3$. Thus $\mathfrak{S}_3/\mathfrak{A}_3 \subset G/H$ switches the fibers of π and $\{\pm 1\} = \ker(\text{pr}_1) \subset G/H$ preserves each fiber of π .

Lemma 3.5. *The action of G on $S_4(V)$ is simply transitive.*

Proof. We use the notation in Remark 3.4. Since the action of \mathfrak{S}_3 on $S_2(V)$ is simply transitive and $\tilde{\pi}$ is compatible with $\text{pr}_1 : G \rightarrow \mathfrak{S}_3$, it is enough to prove that $\mathbb{F}_2^{\mathfrak{S}_3} = \ker(\text{pr}_1)$ acts on each fiber of $\tilde{\pi}$ simply transitively.

To prove this, let us fix an element $(A_1, A_2, A_3) \in S_4(V)$ and consider its fiber F of $\tilde{\pi}$. We make an identification $f : \mathbb{F}_2^{\mathfrak{S}_3} \xrightarrow{\sim} V[2]^3$ by sending $[\sigma]$ to (A'_1, A'_2, A'_3) with $A'_{\sigma(2)} = A_{\sigma(1)}$ and $A'_{\sigma(1)} = A'_{\sigma(3)} = 0$. If we define an action of $V[2]^3$ on F by the componentwise addition, then the action $\mathbb{F}_2^{\mathfrak{S}_3}$ on F is identified via f with the action $V[2]^3$ on it, which obviously acts simply and transitively on F . \square

Corollary 3.6. *We have $\#T_4(V) = 4$.*

Proof. By Lemma 3.11, the action of G/H on $T_4(V)$ is simply transitive. Our assertion follows from this and $\#G/H = 4$. \square

Remark 3.7. Let $[P, Q, R] \in T_4(V)$. Since $G/H = \mathfrak{A}_3 \times \{\pm 1\}$ is generated by $((12), 1)$ and $(id, -1)$, we see that

$$\begin{aligned}T_4(V) &= (G/H) \cdot [P, Q, R] \\ &= \{\pm[P, Q, R], \pm[Q, P, R]\}.\end{aligned}$$

We use the following lemma and remark in the next subsection to define an action of $\bigwedge^2 V$ on $T_4(V)$.

Lemma 3.8. *Let X be a set consisting of 4 elements. Denote by S_1 the set of cyclic subgroup of order 4 in $\text{Aut}(X)$, and by S_2 the set of elements in $\text{Aut}(X)$ of order 2 with no fixed points on X .*

(1) For any $C \in S_1$, the action of C on X is simply transitive. In particular, the order 2 element τ in C belongs to S_2 .

(2) Mapping each $C \in S_1$ to its element of order 2 defines a bijection $d : S_1 \xrightarrow{\sim} S_2$. For any $\tau \in S_2$, we denote $d^{-1}(\tau)$ by $C(\tau)$.

Proof. (1) Considering the orbit decomposition of X by C , we see that the action of C on X is transitive; if not, any element of C must be of order 1 or 2, which is a contradiction. This shows (1) since $\#C = \#X (= 4)$.

(2) By (1), we can define the map d . Note that $\text{Aut}(X)$ acts transitively on both S_1 and S_2 by conjugation, and that d commutes with these actions. Also, $\#S_1 = \#S_2 = 3$, since there are 6 elements of order 4 in $\text{Aut}(X)$ and each $C \in S_1$ has two generators, and since S_2 is identified with the set of 2-2 partitions via the orbit decompositions. These shows that d is bijective. \square

3.3 The action of $\bigwedge^2 V$ on $T_n(V)$

For each basis (P, Q) of V , we denote by $\varphi_{P,Q} \in \mathrm{SL}(V)$ the map determined by $\varphi_{P,Q}(P) = P$ and $\varphi_{P,Q}(Q) = P + Q$.

Lemma 3.9. *There exists a unique surjective homomorphism $\varphi : \bigwedge^2 V \rightarrow \mathrm{SL}(V)^{\mathrm{ab}}$ such that, for any basis (P, Q) of V , φ maps $P \wedge Q$ to $\overline{\varphi_{P,Q}} \in \mathrm{SL}(V)^{\mathrm{ab}}$.*

Proof. For a basis (P, Q) of V , mapping a generator $P \wedge Q$ of $\bigwedge^2 V$ to $\overline{\varphi_{P,Q}} \in \mathrm{SL}(V)^{\mathrm{ab}}$ defines a homomorphism $\varphi : \bigwedge^2 V \rightarrow \mathrm{SL}(V)^{\mathrm{ab}}$. To see that φ is independent of the choice of (P, Q) , we claim that, for $u \in \mathrm{GL}(V)$, we have $\varphi_{P,Q}^{\det u} \equiv \varphi_{u(P),u(Q)}$ in $\mathrm{SL}(V)^{\mathrm{ab}}$. Let v be an element of $\mathrm{GL}(V)$ given by $v(P) = P$ and $v(Q) = (\det u)Q$. Then, we have $\varphi_{P,Q}^{\det u} = \varphi_{P,(\det u)Q} = v\varphi_{P,Q}v^{-1}$. Also, $\varphi_{u(P),u(Q)} = u\varphi_{P,Q}u^{-1}$. Since $u^{-1}v \in \mathrm{SL}(V)$, the claim follows.

By the above claim, we see that φ is surjective as follows. Because $\mathrm{SL}(V)$ is generated by all the elements $\varphi_{u(P),u(Q)} = u\varphi_{P,Q}u^{-1}$ with $u \in \mathrm{GL}(V)$, the claim shows that $\mathrm{SL}(V)^{\mathrm{ab}}$ is generated by $\overline{\varphi_{P,Q}}^{\pm 1}$, and thus by $\varphi(P \wedge Q) = \overline{\varphi_{P,Q}}$. \square

We let $\mathrm{SL}(V)$ canonically act on $T_n(V)$ and consider the corresponding homomorphism $\tilde{\psi} : \mathrm{SL}(V) \rightarrow \mathrm{Aut}(T_n(V))$.

Proposition 3.10. *(1) Let C denote the subgroup $\mathfrak{A}(\mathrm{Aut}(T_3(V)))$ (resp. $C(-1)$ as in Lemma 3.8 (2)) of $\mathrm{Aut}(T_n(V))$ for $n = 3$ (resp. $n = 4$). Then, the map $\tilde{\psi}$ induces an isomorphism*

$$\psi : \mathrm{SL}(V)^{\mathrm{ab}} \xrightarrow{\simeq} C.$$

(2) The surjective map $\varphi : \bigwedge^2 V \rightarrow \mathrm{SL}(V)^{\mathrm{ab}}$ in Lemma 3.9 is an isomorphism.

Proof. (1) (2) We first prove that the map $\tilde{\psi}$ induces a *surjective* homomorphism

$$\psi : \mathrm{SL}(V)^{\mathrm{ab}} \twoheadrightarrow C.$$

Since $\mathrm{SL}(V)$ is generated by the elements $\varphi_{X,Y}$ for all bases (X, Y) of V , it suffices to show that, for a basis (P, Q) of V , $\tilde{\psi}$ maps $\varphi_{P,Q}$ to a generator of C .

For $n = 3$, this follows because $T_3(V) = \{[P, Q], [P, P + Q], [P, 2P + Q]\}$ and $\varphi_{P,Q}([P, iP + Q]) = [P, (i + 1)P + Q]$.

We next consider the case $n = 4$. Since $\varphi_{P,Q}^2[P, Q, P + Q] = [P, Q + 2P, (P + Q) + 2P] = ([\mathrm{id}] + [23])[P, Q, P + Q] = -[P, Q, P + Q]$, we see that $\tilde{\psi}(\varphi_{P,Q})$ is of order 4. Further, by Lemma 3.8, $\tilde{\psi}(\varphi_{P,Q}^2)$ is of order 2 with no fixed points, which implies that $\varphi_{P,Q}^2[Q, P, P + Q]$ must be $-[Q, P, P + Q]$. Therefore, $\tilde{\psi}(\varphi_{P,Q}^2) = -1$ and $C(-1)$ is generated by $\varphi_{P,Q}$.

The surjective maps ψ and φ are actually bijective because $\# \bigwedge^2 V = \# C (= 4)$. \square

We now define an action of $\bigwedge^2 V$ on $T_n(V)$ by the composition

$$\psi \circ \varphi : \bigwedge^2 V \xrightarrow{\simeq} \mathrm{SL}(V)^{\mathrm{ab}} \xrightarrow{\simeq} C \subset \mathrm{Aut}(T_n(V)).$$

Corollary 3.11. *The action of $\bigwedge^2 V$ on $T_n(V)$ is simply transitive.*

Proof. This follows from Remark 3.2, Lemma 3.8 (1), and Proposition 3.10. \square

Next, we see that the action $\bigwedge^2 V \curvearrowright T_n(V)$ has a simple description when V is moreover accompanied with an extension

$$0 \rightarrow L \rightarrow V \xrightarrow{p} \mathbb{Z}/(n) \rightarrow 0 \quad (3.1)$$

of $\mathbb{Z}/(n)$. We observe the following:

(1) Denote $p^{-1}(1)$ by T . Then, mapping each $P \in L$ to $P \wedge Q$ with $Q \in T$ defines an isomorphism $\epsilon : L \xrightarrow{\sim} \bigwedge^2 V$, which is independent of the choice of $Q \in T$

(2) Corresponding to the above extension, we let L act on V by an injective homomorphism $\tilde{\varphi} : L \hookrightarrow \mathrm{SL}(V)$ defined by $\tilde{\varphi}(P)(Q) := Q + p(Q)P$ for $P \in L$ and $Q \in V$. Note that, if we define a subgroup M of $\mathrm{SL}(V)$ by

$$M = \{\sigma \in \mathrm{SL}(V) : \sigma(P) = P \text{ for all } P \in L\},$$

then $\tilde{\varphi}$ induces an isomorphism $\tilde{\varphi} : L \xrightarrow{\sim} M$, whose inverse $f : M \rightarrow L$ is obviously given by $f(\sigma) := \sigma(Q) - Q$ with any $Q \in T$. We also remark that the action $L \simeq M \curvearrowright V$ preserves T , which coincides with the canonical action by translations; in particular, this action of L on T is obviously simply transitive.

Example 3.12. (1) If $V = E[n]$ for a Tate curve E , then we have a canonical extension (2.6) in Theorem 2.9.

(2) If we fix a basis (P, Q) of V , then we have an extension

$$0 \rightarrow \langle P \rangle \rightarrow V \xrightarrow{p} \mathbb{Z}/(n) \rightarrow 0, \quad (3.2)$$

where $p : V \rightarrow \mathbb{Z}/(n)$ is given by $p(P) = 0$ and $p(Q) = 1$.

Remark 3.13. (1) Any extension (3.1) can become of the form (3.2) by taking a basis (P, Q) with P a generator of L and $Q \in T$.

(2) For any extension (3.1), the following diagram is commutative:

$$\begin{array}{ccc} L & \xrightarrow{\epsilon} & \bigwedge^2 V \\ \tilde{\varphi} \downarrow & & \varphi \downarrow \\ M & \xrightarrow{\text{quotient}} & \mathrm{SL}(V)^{\mathrm{ab}}. \end{array}$$

To see this, we may assume that (3.1) is of the form (3.2) by (1). Then, we obtain $\tilde{\varphi}(P)(Q) = \varphi_{P,Q}$ and $\epsilon(P) = P \wedge Q$ so that $\varphi(\epsilon(P)) = \overline{\varphi_{P,Q}} \in \mathrm{SL}(V)^{\mathrm{ab}}$.

For the given extension (3.1), we define a map $\tau : T \rightarrow T_n(V)$ by $\tau(Q) = [P, Q]$ (resp. $\tau(Q) = [Q, P, -(P + Q)]$) for $n = 3$ (resp. $n = 4$) with P a generator of L .

Lemma 3.14. (1) The map τ is independent of the choice of P .

(2) The map τ is compatible with $\epsilon : L \xrightarrow{\sim} \bigwedge^2 V$.

(3) The map τ is bijective.

Proof. (1) The other generator of L is $-P$ for both cases $n = 3$ and $n = 4$. If $n = 3$, then it defines the same point in $\mathbb{P}(V)$ as P . If $n = 4$, then we see that $[Q, -P, -(-P + Q)] = ([123] + [id] + [13])[Q, P, -(P + Q)] = [Q, P, -(P + Q)]$ for $Q \in T$. These prove (1).

(2) First, note that τ is tautologically compatible with $\tilde{\varphi} : L \rightarrow \mathrm{SL}(V)$. By Proposition 3.10 (1), it is compatible with the composite map $L \xrightarrow{\tilde{\varphi}} \mathrm{SL}(V) \rightarrow \mathrm{SL}(V)^{\mathrm{ab}}$, which coincides with the composite map $L \xrightarrow{\epsilon} \bigwedge^2 V \xrightarrow{\varphi} \mathrm{SL}(V)^{\mathrm{ab}}$ by Remark 3.13 (2). Thus (2) follows, since the action of $\bigwedge^2 V$ on $T_n(V)$ is defined via $\varphi : \bigwedge^2 V \xrightarrow{\sim} \mathrm{SL}(V)^{\mathrm{ab}}$.

(3) Because the case $n = 3$ is obvious from the definition of τ , we show the case $n = 4$. By (2), we have an action of L on $T' := \tau(T)$, which is transitive because the action of L on T is transitive. The order 2 element A of L acts on T' as -1 ; in fact, by Remark 3.13, we may assume that the given extension is of the form (3.2), in which case $A = 2P$ acts on T' as -1 as shown in Proposition 3.10. Since -1 is fixed point free on $T_4(V)$, the stabilizer of any element of T' in L has no elements of order 2; that is, it is trivial. This shows that $\#T' = \#T_4(V) (= 4)$, and hence τ is bijective. \square

Remark 3.15. *We can also deduce Corollary 3.11 from Lemma 3.14 as follows. Fix an extension (3.1). By Lemma 3.14, we identify the action $\bigwedge^2 V \curvearrowright T_n(V)$ with the action $L \curvearrowright T$, which is obviously simply transitive,*

4 A bijection $w_n : T_n(E[n]) \rightarrow \mu_n \sqrt[n]{\Delta_E}$

Here, let n be 3 or 4, and E be an elliptic curve given by a Weierstrass equation over a field K of characteristic prime to n . We apply the construction in Section 3 to $V = E[n]$, and construct a bijection $w_n : T_n(E[n]) \rightarrow \mu_n \sqrt[n]{\Delta_E}$.

4.1 The case $n = 3$

The result in this section is based on 5.5, [7]. Fix a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad (4.1)$$

for E , and denote its discriminant by Δ . Let b_i ($i = 2, 4, 6, 8$) be the constants defined in Definition 2.3. For a point P on E , we denote by (x_P, y_P) its coordinate in terms of this Weierstrass equation, and by Δ the discriminant of it. If P is a point of order 3, then P satisfies $x_P = x_{-P} = x_{2P}$. Since

$$x_{2P} = \frac{x_P^4 - b_4x_P^2 - 2b_6x_P - b_8}{4x_P^3 + b_2x_P^2 + 2b_4x_P + b_6}$$

(III, 2.3, [8].), the x -coordinate of any point of order 3 is a solution of the equation

$$x = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6};$$

that is,

$$x^4 + \frac{b_2}{3}x^3 + b_4x^2 + b_6x + \frac{b_8}{3} = 0. \quad (4.2)$$

For $\{X, Y\} \in T_3(E[3])$, let x_1 and x_2 be the x -coordinates of the elements in X , and x_3 and x_4 be the x -coordinates of the elements in Y . Using this, we define

$$w_3(\{X, Y\}) := b_4 - 3(x_1x_2 + x_3x_4),$$

which is obviously G_K -equivariant.

Lemma 4.1. *Let the notation be as above. Then,*

$$T^3 - \Delta = \prod (T - w_3(\{X, Y\})), \quad (4.3)$$

where the product is taken over $T_3(E[3])$. In other words, w_3 defines a bijection $w_3 : T_3(E[3]) \rightarrow \mu_3 \sqrt[3]{\Delta_E}$.

Proof. We first show our assertion when E is defined over a field K and (4.1) is of the Deuring normal form

$$y^2 + \alpha xy + y = x^3 \quad (\alpha \in \bar{K}, \alpha^3 \neq 27).$$

In this case, we have

$$b_2 = \alpha^2, b_4 = \alpha, b_6 = 1, b_8 = 0, \text{ and } \Delta = \alpha^3 - 27.$$

Then, (4.2) is

$$X^4 + \frac{1}{3}\alpha^2 X^3 + \alpha X^2 + X = 0, \quad (4.4)$$

and the solutions of (4.4) are just the x -coordinates x_1, \dots, x_4 of $E[3]$; one of which is 0, say $x_4 = 0$. It follows that

$$X^3 + \frac{1}{3}\alpha^2 X^2 + \alpha X + 1 = (X - x_1)(X - x_2)(X - x_3). \quad (4.5)$$

Note that, under our present assumption, (4.3) becomes

$$T^3 - (\alpha^3 - 27) = \prod_{1 \leq i < j \leq 3} (T - \alpha + 3x_i x_j).$$

Replacing $Y = T - \alpha$ reduces the problem to showing that

$$Y^3 + 3\alpha Y^2 + 3\alpha^2 Y + 27 = (Y + 3x_1x_2)(Y + 3x_2x_3)(Y + 3x_1x_3). \quad (4.6)$$

This is true from the following computation: Setting $X = 3/Y$ in (4.5) and using $x_1x_2x_3 = -1$, we see that (4.5) becomes

$$\begin{aligned} \frac{1}{Y^3}(27 + 3\alpha^2 Y + 3\alpha Y^2 + Y^3) &= \frac{1}{Y^3}(3 - x_1 Y)(3 - x_2 Y)(3 - x_3 Y) \\ &= -\frac{x_1 x_2 x_3}{Y^3}(-3x_1^{-1} + Y)(-3x_2^{-1} + Y)(-3x_3^{-1} + Y) \\ &= \frac{1}{Y^3}(3x_2x_3 + Y)(3x_1x_3 + Y)(3x_1x_2 + Y), \end{aligned}$$

which is none other than (4.6).

Next, we prove the general case. Suppose we are given two Weierstrass equations for E over K with coordinates (x, y) and (x', y') . We use the notation in Remark 2.5. (1). By (2.1) and (2.3), we obtain

$$b_4 - 3(x_i x_j + x_k x_l) = u^4(b'_4 - 3(x'_i x'_j + x'_k x'_l)). \quad (4.7)$$

By (2.3) and (4.7), it follows that (4.3) for (x, y) is equivalent to (4.3) for (x', y') . Then, the assertion follows from the fact that any elliptic curve over K has a Weierstrass equation of the Deuring normal form over \bar{K} . (This requires $\text{char}(K) \neq 3$. For the detail, see Proposition 1.3, Appendix A, [8].) \square

Let us check how the map w_3 changes if we take another Weierstrass equation for E . Let the notation be as in Remark 2.5 (1), and denote by w'_3 the map as above for another Weierstrass equation for E with the coordinate (x', y') .

Lemma 4.2. *We have $w_3 = u^4 w'_3$.*

Proof. Let (P, Q) be a basis of $E[3]$. Then x_P, x_Q, x_{P+Q} , and x_{P-Q} are all distinct, and they must satisfy the above equation. Considering the coefficients of x^3 in (4.2), we obtain

$$x_P + x_Q + x_{P+Q} + x_{P-Q} = -\frac{b_2}{3}.$$

Using this equality and the relations in Remark 2.5 (1), we obtain $w_3([P, Q]) = u^4 w'_3([P, Q])$. \square

Lemma 4.2 and $\Delta = u^{12} \Delta'$ imply that the diagram

$$\begin{array}{ccc} & & \mu_3 \sqrt[3]{\Delta} \\ & \nearrow^{w_3} & \uparrow^{u^4 \times} \\ T_3(E[3]) & & \\ & \searrow_{w'_3} & \downarrow \\ & & \mu_3 \sqrt[3]{\Delta'} \end{array}$$

is commutative. Since the vertical map $u^4 \times$ is an isomorphism of μ_3 -torsors over k , we identify w_3 and w'_3 , and we consider them as a map from $T_3(E[3])$ to the μ_3 -torsor $\mu_3 \sqrt[3]{\Delta_E}$. We denote this map again by w_3 .

Remark 4.3. The construction of $T_3(E[3])$ and w_3 are easily generalized to the case that E is an elliptic curve over a scheme S with 3 invertible. This is done by considering the locally constant étale sheaf $E[3]$ over S and by étale descent.

4.2 The case $n = 4$

Since K is of characteristic prime to 2, we can take a Weierstrass equation for E of the form

$$y^2 = x^3 + a_2 x^2 + a_4 x + a_6.$$

In this subsection, we only consider Weierstrass equations of this form. In order to write a 4-th root of Δ_E in terms of $E[4]$, first we prove the following lemma. (See also [4, §11].)

Lemma 4.4. *Let the notation be as above. Let $A \in E$ be a point of order 2, and P and P' be points of E satisfying $2P = 2P' = A$ and $P \neq \pm P'$. Then we have*

$$\left(\frac{y_P - y_{P'}}{x_P - x_{P'}} \right)^2 = x_A - x_{P-P'}.$$

Proof. Denote B and C for the points of order 2 on E other than A . Then, note that

$$x^3 + a_2x^2 + a_4x + a_6 = (x - x_A)(x - x_B)(x - x_C),$$

and in particular we have $x_A + x_B + x_C = -a_2$. Since $P + P' + (-P - P') = O$, the three points P, P' , and $-P - P'$ are colinear. So the solutions for the equation

$$\left(\left(\frac{y_P - y_{P'}}{x_P - x_{P'}} \right) (x - x_P) + y_P \right)^2 = x^3 + a_2x^2 + a_4x + a_6$$

are exactly $x_P, x_{P'}$, and $x_{-P-P'} (= x_{P+P'})$. Considering the coefficients of x^2 , we have

$$x_P + x_{P'} + x_{P+P'} = \left(\frac{y_P - y_{P'}}{x_P - x_{P'}} \right)^2 - a_2. \quad (4.8)$$

On the other hand, we show

$$x_P + x_{P'} = 2x_A. \quad (4.9)$$

To prove (4.9), let T be a 4-division point of E with $2T = A$. Then the same argument as above shows that

$$2x_T + x_A = \left(\frac{y_T - y_A}{x_T - x_A} \right)^2 - a_2. \quad (4.10)$$

Combining (4.10) with

$$\begin{aligned} (y_T - y_A)^2 &= y_T^2 \\ &= (x_T - x_A)(x_T - x_B)(x_T - x_C), \end{aligned}$$

we have the following quadratic equation for x_T :

$$x_T^2 - (-a_2 + x_A - x_B - x_C)x_T - (x_A^2 + a_2x_A + x_Bx_C) = 0.$$

Since the solutions for this equation are exactly x_P and $x_{P'}$, we obtain

$$\begin{aligned} x_P + x_{P'} &= -a_2 + x_A - x_B - x_C \\ &= 2x_A, \end{aligned}$$

and hence (4.9) follows.

Now, combining (4.8) and (4.9), we obtain

$$x_A + (x_A + x_{P+P'}) = \left(\frac{y_P - y_{P'}}{x_P - x_{P'}} \right)^2 - a_2. \quad (4.11)$$

Since

$$\{A, P + P', P - P'\} = E[2] \setminus O,$$

we have

$$x_A + x_{P+P'} + x_{P-P'} = -a_2. \quad (4.12)$$

Then, (4.11) and (4.12) show the lemma. \square

An immediate corollary of Lemma 4.4 is the following:

Corollary 4.5. For $(P, Q, R) \in S_4(E[4])$, set

$$P' = P + 2Q, Q' = Q + 2R, R' = R + 2P.$$

Then

$$\tilde{w}_4(P, Q, R) := 2 \frac{y_P - y_{P'}}{x_P - x_{P'}} \cdot \frac{y_Q - y_{Q'}}{x_Q - x_{Q'}} \cdot \frac{y_R - y_{R'}}{x_R - x_{R'}}$$

is a 4-th root of Δ , and the map $\tilde{w}_4 : S_4(E[4]) \rightarrow \mu_4 \sqrt[4]{\Delta}$ makes the following diagram commutative:

$$\begin{array}{ccc} S_4(E[4]) & \xrightarrow{\tilde{w}_4} & \mu_4 \sqrt[4]{\Delta} \\ \downarrow \tilde{\pi} & & \downarrow \text{squaring} \\ S_2(E[2]) & \xrightarrow{\tilde{w}_2} & \pm \sqrt{\Delta}, \end{array}$$

where $\tilde{w}_2 : S_2(E[2]) \rightarrow \pm \sqrt{\Delta}$ is a canonical bijection given by

$$\tilde{w}_2(A, B, C) := 4(x_A - x_B)(x_B - x_C)(x_C - x_A).$$

Remark 4.6. Let the notation be as in Corollary 4.5. Then, we see that

$$\begin{aligned} \tilde{w}_4(-P, Q, R) &= -\tilde{w}_4(P, Q, R) \\ \tilde{w}_4(P', Q, R) &= \tilde{w}_4(P, Q, R) \end{aligned}$$

for $(P, Q, R) \in S_4(E[4])$. We also have the analogous properties for Q and R .

Suppose that we are given two Weierstrass equations defining E with coordinates (x, y) and (x', y') . Denote the above maps by \tilde{w}_4 and \tilde{w}'_4 , and their discriminants by Δ_4 and Δ'_4 , respectively.

Lemma 4.7. We have $\tilde{w}_4 = u^3 \tilde{w}'_4$.

Proof. This is obvious from Remark 2.5 (2). \square

Lemma 4.7 and $u^{12} \Delta' = \Delta$ give the following commutative diagram:

$$\begin{array}{ccc} & & \mu_4 \sqrt[4]{\Delta} \\ & \nearrow \tilde{w}_4 & \uparrow u^3 \times \\ S_4(E[4]) & & \mu_4 \sqrt[4]{\Delta'} \\ & \searrow \tilde{w}'_4 & \end{array}$$

By the same reason explained at the end of the last subsection, we identify \tilde{w}_4 and \tilde{w}'_4 , and we consider them as a map from $S_4(E[4])$ to the μ_4 -torsor $\mu_4 \sqrt[4]{\Delta_E}$. Denote this map simply by \tilde{w}_4 .

Lemma 4.8. The map $\tilde{w}_4 : S_4(E[4]) \rightarrow \mu_4 \sqrt[4]{\Delta_E}$ factors through the quotient $T_4(E[4])$.

Proof. We use the notation in Subsection 3.3 and Corollary 4.5. It is obvious that \tilde{w}_4 is invariant under the action of \mathfrak{A}_3 . We claim that for each $(P, Q, R) \in S_4(E[4])$ we have

$$\tilde{w}_4([\sigma](P, Q, R)) = \begin{cases} -\tilde{w}_4(P, Q, R) & (\text{if } \sigma \in \mathfrak{S}_3 \text{ is even}) \\ \tilde{w}_4(P, Q, R) & (\text{if } \sigma \in \mathfrak{S}_3 \text{ is odd}). \end{cases} \quad (4.13)$$

Since every $[\sigma]$ for even (resp. odd) σ is conjugate to $[\text{id}]$ (resp. to $[(12)]$) by an element in $\mathfrak{A}_3 \subset \text{Aut}(S(E[4]))$, it is enough to check the claim for $[\text{id}]$ and $[(12)]$. In these cases, we see that

$$\begin{aligned} \tilde{w}_4([\text{id}](P, Q, R)) &= \tilde{w}_4(P, -Q', R) \\ &= -\tilde{w}_4(P, Q, R) \\ \tilde{w}_4([(12)](P, Q, R)) &= \tilde{w}_4(P', Q, R) \\ &= \tilde{w}_4(P, Q, R) \end{aligned}$$

by Remark 4.6. □

Lemma 4.8 gives us a map

$$\begin{aligned} w_4 : T_4(E[4]) &\longrightarrow \mu_4 \sqrt[4]{\Delta_E} \\ [P, Q, R] &\mapsto \tilde{w}_4(P, Q, R) \end{aligned}$$

induced from \tilde{w}_4 , which is G_K -equivariant as in the case $n = 3$.

Remark 4.9. The commutative diagram in Corollary 4.5 induces a commutative diagram

$$\begin{array}{ccc} T_4(E[4]) & \xrightarrow{w_4} & \mu_4 \sqrt[4]{\Delta} \\ \downarrow \pi & & \downarrow \text{squaring} \\ T_2(E[2]) & \xrightarrow{w_2} & \pm \sqrt{\Delta} \end{array}$$

of G_K -sets. Here, w_2 is the map induced from \tilde{w}_2 . By Remark 4.6, w_4 is compatible with $\text{pr}_2 : G/H \rightarrow \{\pm 1\}$.

Corollary 4.10. *The map $w_4 : T_4(E[4]) \rightarrow \mu_4 \sqrt[4]{\Delta_E}$ is bijective.*

Proof. Fix an element $X \in T_4(E[4])$. Since w_4 is compatible with $\text{pr}_2 : G/H \rightarrow \{\pm 1\}$,

$$\begin{aligned} w_4(T_4(E[4])) &= w_4((G/H) \cdot X) \\ &= \{\pm w_4(X), \pm w(\sigma X)\} \end{aligned}$$

for any odd permutation $\sigma \in \mathfrak{S}_3$. The commutativity of the diagram in Remark 4.11 shows that $\pm w_4(X)$ and $\pm w(\sigma X)$ belong to distinct fibers of the squaring map $\mu_4 \sqrt[4]{\Delta_4} \rightarrow \pm \sqrt{\Delta_4}$. Thus, we obtain $\#w_4(T_4(E[4])) = 4$ and hence w_4 is bijective. □

Remark 4.11. For the same reason as in Remark 4.3, we can define $T_4(E[4])$ and w_4 for an elliptic curve over a scheme on which 4 is invertible.

4.3 The Tate curve case

Let $E = E_q$ be the Tate curve over the field $K = \mathbb{Q}((q))$ of Laurent series. By Theorem 2.9 (1), $E[n]$ is canonically an extension of $\mathbb{Z}/(n)$ in the category of G_K -modules:

$$0 \rightarrow \mu_n \rightarrow E[n] \rightarrow \mathbb{Z}/(n) \rightarrow 0.$$

For this extension, we use the notations and results in Subsection 3.3 by setting $V = E[n]$. In our Tate curve case, we identify

$$\begin{aligned} L &= \mu_n, \\ T &= \mu_n \sqrt[n]{q}, \text{ and} \\ \epsilon &= e_n^{-1} : L \xrightarrow{\sim} \bigwedge^2 E[n], \end{aligned}$$

the last equality of which is due to Remark 2.10.

Note that the action of G_K on $E[n]$ gives rise to an isomorphism

$$g : \text{Gal}(K(\mu_n, \sqrt[n]{q})/K(\mu_n)) \xrightarrow{\sim} M.$$

We also remark that the composite map $f \circ g : \text{Gal}(K(\mu_n, \sqrt[n]{q})/K(\mu_n)) \xrightarrow{\sim} M \xrightarrow{\sim} L = \mu_n$ is the Kummer character h_1 mapping $\sigma \in \text{Gal}(K(\mu_n, \sqrt[n]{q})/K(\mu_n))$ to $\sigma(\sqrt[n]{q})/\sqrt[n]{q}$.

Proposition 4.12. *The composition $w_n \circ \tau : T \rightarrow \mu_n \sqrt[n]{\Delta_E}$ is an isomorphism of $L = \mu_n$ -torsors over K .*

Proof. It is obvious from the constructions that $w_n \circ \tau$ is G_K -equivariant. We show that it is also μ_n -equivariant. By definition, the action of L on T is identified with the action of M on T via the isomorphism $\tilde{\varphi} : L \xrightarrow{\sim} M$, which is also identified with the action of $\text{Gal}(K(\mu_n, \sqrt[n]{q})/K(\mu_n))$ on T via $g : \text{Gal}(K(\mu_n, \sqrt[n]{q})/K(\mu_n)) \xrightarrow{\sim} M$. On the other hand, the action of G_K on $\mu_n \sqrt[n]{\Delta}$ induces an action of $\text{Gal}(K(\mu_n, \sqrt[n]{q})/K(\mu_n))$ on $\mu_n \sqrt[n]{\Delta}$, which is identified with a canonical action of μ_n on $\mu_n \sqrt[n]{\Delta}$ via the Kummer character $h_2 : \text{Gal}(K(\mu_n, \sqrt[n]{q})/K(\mu_n)) \xrightarrow{\sim} \mu_n$ mapping σ to $\sigma(\sqrt[n]{\Delta})/\sqrt[n]{\Delta}$. Since $w_n \circ \tau$ is G_K -equivariant, it is also $\text{Gal}(K(\mu_n, \sqrt[n]{q})/K(\mu_n))$ -equivariant. These arguments imply that the map $w_n \circ \tau : T \rightarrow \mu_n \sqrt[n]{\Delta_E}$ is compatible with $h_2 \circ (f \circ g)^{-1} = h_2 \circ h_1^{-1} : L \rightarrow \mu_n$, which is the identity map since $\sqrt[n]{\Delta}/\sqrt[n]{q} \in K(\mu_n)^\times$ by (2.3). Therefore, $w_n \circ \tau$ is $L = \mu_n$ -equivariant. \square

Corollary 4.13. *The map w_n is compatible with the normalized Weil pairing e_n .*

Proof. This immediately follows from Corollary 3.14 (2) (3), Proposition 4.12, and $e_n = \epsilon^{-1}$. \square

Next we consider the following canonical isomorphism

$$\begin{aligned} \delta : T &\rightarrow \mu_n \sqrt[n]{\Delta_E} \\ z &\mapsto z \prod_{m \geq 1} (1 - q^m)^{24/n} \end{aligned}$$

of $L = \mu_n$ -torsors over K .

Proposition 4.14. *We have $w_n \circ \tau = \delta$.*

Proof. By Proposition 4.12, the composite map $\sigma := (w_n \circ \tau) \circ \delta^{-1}$ is an automorphism of the μ_n -torsor $\mu_n \sqrt[n]{\Delta_E}$ over K , and hence σ belongs to $\mu_n(K) \subset \text{Aut}(\mu_n \sqrt[n]{\Delta_E})$. Because $\mu_n(K) = \{1\}, \{\pm 1\}$ for $n = 3, 4$ respectively, the assertion for $n = 3$ is proved, but there remains a possibility of $w_4 \circ \tau = -\delta$.

To determine the sign of σ for $n = 4$, we check that the first coefficient of $w_4(\tau(z))$, which belongs to $\mathbb{Q}(i)((z))$, coincides with that of $\delta(z) = z \prod_{m \geq 1} (1 - q^m)^6$; that is, z . Note that, if $(P, Q, R) = \tau(z) = (z, i, (iz)^{-1})$ and if P', Q', R' are as in Corollary 4.5, then we obtain $(P', Q', R') = (-z, -iz^2, -iz)$.

Change variables of E as

$$X = x \quad \text{and} \quad Y = y + \frac{1}{2}x$$

to make the Weierstrass equation of E into the form

$$Y^2 = X^3 + A_2X^2 + A_4X + A_6.$$

Then, it follows from Theorem 2.9 that

$$\begin{aligned} \bar{K}^\times / q^\mathbb{Z} &\xrightarrow{\cong} \left\{ \begin{array}{l} \bar{K}\text{-valued points on} \\ y^2 + xy = x^3 + a_4x + a_6 \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{l} \bar{K}\text{-valued points on} \\ Y^2 = X^3 + A_2X^2 + A_4X + A_6 \end{array} \right\} \\ u &\longmapsto (x(u, q), y(u, q)) \longmapsto (x(u, q), y(u, q) + \frac{1}{2}x(u, q)) \end{aligned}$$

We have

$$x(u) := x(u, z^4) = f(u) + \sum_{n \geq 1} \left(\frac{z^{4n}u}{(1 - z^{4n}u)^2} + \frac{z^{4n}u^{-1}}{(1 - z^{4n}u^{-1})^2} - 2 \frac{nz^{4n}}{1 - z^{4n}} \right),$$

where $f(u) = \frac{u}{(1-u)^2} \equiv u + 2u^2 \pmod{z^3}$. Also, set

$$\begin{aligned} Y(u) &:= y(u, z^4) + \frac{1}{2}x(u, z^4) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \frac{z^{4n}u(1 + z^{4n}u)}{(1 - z^{4n}u)^2} \\ &= g(u) + \frac{1}{2} \sum_{n \geq 1} \left(\frac{z^{4n}u^{-1}(z^{4n}u^{-1} + 1)}{(z^{4n}u^{-1} - 1)^3} + \frac{z^{4n}u(1 + z^{4n}u)}{(1 - z^{4n}u)^3} \right), \end{aligned}$$

where $g(u) = u(1+u)/2(1-u)^3 \equiv \frac{1}{2}u + 2u^2 \pmod{z^3}$. Note that the second term of $x(u)$ and $Y(u)$ is congruent to 0 $\pmod{z^2}$ whenever the degree of $u \in \mathbb{Q}(i)((z))$ in z is of 0, $\pm 1, \pm 2$, and that $f(u) = f(u^{-1})$ and $g(u) = -g(u^{-1})$. Then, the computation goes as follows:

$$\begin{aligned} w_4([z, i, iz]) &= 2 \frac{Y(z) - Y(-z)}{x(z) - x(-z)} \cdot \frac{Y(i) - Y(-iz^2)}{x(i) - x(-iz^2)} \cdot \frac{Y((iz)^{-1}) - Y(-iz)}{x((iz)^{-1}) - x(-iz)} \\ &\equiv 2 \frac{g(z) - g(-z)}{f(z) - f(-z)} \cdot \frac{g(i) - g(-iz^2)}{f(i) - f(-iz^2)} \cdot \frac{g((iz)^{-1}) - g(-iz)}{f((iz)^{-1}) - f(-iz)} \pmod{z^2}. \end{aligned} \tag{4.14}$$

For each factor in the right hand side of (4.14), we see that

$$\begin{aligned} \frac{g(z) - g(-z)}{f(z) - f(-z)} &\equiv \frac{z}{2z} \pmod{z^2} \\ &\equiv \frac{1}{2} \pmod{z^2}, \end{aligned}$$

$$\begin{aligned}\frac{g(i) - g(-iz^2)}{f(i) - f(-iz^2)} &\equiv \frac{g(i)}{f(i)} \pmod{z^2} \\ &\equiv \frac{i}{2} \pmod{z^2},\end{aligned}$$

and

$$\begin{aligned}\frac{g((iz)^{-1}) - g(-iz)}{f((iz)^{-1}) - f(-iz)} &= \frac{-g(iz) - g(-iz)}{f(iz) - f(-iz)} \\ &\equiv \frac{2z}{i} \pmod{z^3}.\end{aligned}$$

Therefore,

$$\begin{aligned}w_4([z, i, (iz)^{-1}]) &\equiv 2 \cdot \frac{1}{2} \cdot \frac{i}{2} \cdot \frac{2z}{i} \pmod{z^2} \\ &\equiv z \pmod{z^2}.\end{aligned}$$

Therefore, we have $w_4([z, i, (iz)^{-1}]) = z \prod_{m \geq 1} (1 - q^m)^6$. \square

5 The main theorem

Combining the results of the previous two sections, we now state our main theorem. In the following statement, we consider $T_n(E[n])$ as a μ_n -torsor via the action of $\bigwedge^2 V$ on $T_n(E[n])$ by the identification $e_n : \bigwedge^2 V \xrightarrow{\sim} \mu_n$ with e_n the normalized Weil pairing.

Theorem 5.1. *Let n be 3 or 4.*

(1) *The family of maps*

$$(w_n : T_n(E[n]) \longrightarrow \mu_n \sqrt[n]{\Delta_E})_{E/S}$$

defined in Section 3 with E an elliptic curve over a scheme S on which n is invertible satisfies the following properties:

- (a) *Each map w_n is an isomorphism of μ_n -torsors over S .*
- (b) *The maps w_n 's are compatible with arbitrary base change.*
- (c) *If $E = E_q$ is the Tate curve over $K = \mathbb{Q}((q))$, then the map w_n coincides with $\delta \circ \tau^{-1}$.*

(2) *If $n = 3$, then there exists a unique family of maps $(T_3(E[n]) \xrightarrow{\sim} \mu_3 \sqrt[3]{\Delta_E})_{E/S}$ satisfying (a) and (b) with E an elliptic curve over a scheme S with 3 invertible. Hence it coincides with $(w_3 : T_3(E[3]) \rightarrow \mu_3 \sqrt[3]{\Delta_E})_{E/S}$ and automatically satisfies the property (c).*

(3) *If $n = 4$, then $(w_4)_{E/S}$ and $(-w_4)_{E/S}$ are the only families of maps $T_n(E[4]) \rightarrow \mu_4 \sqrt[4]{\Delta_E}$ satisfying (a) and (b) with E an elliptic curve over a scheme S with 4 invertible. Also, $(w_4)_{E/S}$ is the only family which satisfies the properties (a), (b), and (c).*

Proof. (1) Since (b) is obvious from our construction of w_n and (c) follows from Proposition 4.14, we have only to show (a) for w_n . To do this, we may assume that $E[n]$ (and hence μ_n) is constant over S . Let c_n be the conjugation

$$\text{Aut}(T_n(E[n])) \xrightarrow{\sim} \text{Aut}(\mu_n \sqrt[n]{\Delta_E})$$

by w_n . We consider $\bigwedge^2 E[n]$ as a subgroup of $\text{Aut}(T_n(E[n]))$ by (3.1) and (3.3). We also consider μ_n as a subgroup of $\text{Aut}(\mu_n \sqrt[n]{\Delta_E})$ by the canonical action of μ_n on $\mu_n \sqrt[n]{\Delta_E}$. For the proof, we first show the following lemma:

Lemma 5.2. (1) *The restriction of the map c_n to $\bigwedge^2 E[n]$ induces an isomorphism $e'_n : \bigwedge^2 E[n] \xrightarrow{\sim} \mu_n$.*

(2) *The isomorphism $e'_n : \bigwedge^2 E[n] \xrightarrow{\sim} \mu_n$ is the unique map with which w_n is compatible.*

Proof. (1) If $n = 3$, then the claim follows from the fact that $\text{Aut}(T_3(E[3])) \simeq \text{Aut}(\mu_3 \sqrt[3]{\Delta_E}) \simeq \mathfrak{S}_3$ has the unique subgroup of order 3; the alternating group. If $n = 4$, by Proposition 3.10, we have $\bigwedge^2 E[4] = C(-1) \subset \text{Aut}(T_4(E[4]))$ and $\mu_4 = C(-1) \subset \text{Aut}(\mu_4 \sqrt[4]{\Delta_E})$. Thus, it suffices to check that c_4 maps $-1 \in \text{Aut}(T_4(E[4]))$ to $-1 \in \text{Aut}(\mu_4 \sqrt[4]{\Delta_E})$; that is, $w_4(-X) = -w_4(X)$ for $X \in T_4(E[4])$. This follows from Remark 4.6.

(2) The compatibility is obvious from the definition, and the uniqueness follows because $\mu_n \rightarrow \text{Aut}(\mu_n \sqrt[n]{\Delta_E})$ is injective. \square

We continue the proof of the theorem. To show the claim (1), we check that $e_n = e'_n$.

Denote $A := \Gamma(Y(n), \mathcal{O})$ and let $(E_0/A, (P_0, Q_0))$ be the universal object of $\mathcal{M}(n)$ (see Theorem 2.6). Then the point $e'_n(P_0 \wedge Q_0) \in \mu_n(A)$ defines a morphism $Y(n) \rightarrow \mu_n = \text{Spec}[X, 1/n](X^n - 1)$ of schemes over $\mathbb{Z}[1/n]$, which we also denote by e'_n . This morphism satisfies the following property: If K is a field and $x = (E/K, (P, Q)) \in Y(n)(K)$, then $e'_n(x) = e'_n(P \wedge Q)$. In the same way, the point $e_n(P_0 \wedge Q_0) \in \mu_n(A)$ also defines a morphism $e_n : Y(n) \rightarrow \mu_n$ of schemes over $\mathbb{Z}[1/n]$, which satisfies $e_n(x) = e_n(P \wedge Q)$ for any x as above.

Since $Y(n)$ is connected by Theorem 2.6, the morphism $e_n/e'_n : Y(n) \rightarrow \mu_n$ factors through a connected component U of the scheme μ_n over $\mathbb{Z}[1/n]$. Corollary 4.13 and Lemma 5.2 (2) imply that $e_n/e'_n(E_{z^n}, (\zeta_n, z)) = 1$ for the Tate curve $E_{z^n}/\mathbb{Q}(\zeta_n)((z))$. Therefore, U must be the connected component $\mu_1 = \text{Spec}[X, 1/n]/(X - 1)$ of μ_n . This completes the proof of (1).

(2), (3) Suppose that we are given two maps $W_i : T_n(E[n]) \rightarrow \mu_n \sqrt[n]{\Delta_E}$ ($i = 1, 2$) for each elliptic curve E/S , which satisfy the condition (a) and (b) in the theorem. Then W_1/W_2 defines a morphism $W : \mathcal{M} \rightarrow \mu_n$ of functors $\text{Sch}/\mathbb{Z}[1/n] \rightarrow \text{Sets}$, where \mathcal{M} is the functor defined in Theorem 2.6 (1). By Theorem 2.6 (1), there exists a unique morphism $W' : \mathbb{A}_{\mathbb{Z}[1/n]}^1 \rightarrow \mu_n$ of schemes over $\mathbb{Z}[1/n]$ satisfying $W' \circ j = W : \mathcal{M} \rightarrow \mu_n$, where $j : \mathcal{M} \rightarrow \mathbb{A}_{\mathbb{Z}[1/n]}^1$ is given by taking the j invariant. The morphism W' corresponds to an element of $\mu_n(\mathbb{Z}[1/n, j]) = 1, \{\pm 1\}$ for $n = 3, 4$, respectively. This proves the assertions. \square

Remark 5.3. When $\text{char}(k) \nmid 12$, we define $T_{12}(E[12]) = T_3(E[3]) \times T_4(E[4])$. Then, $T_{12}(E[12])$ admits an action of $\bigwedge^2 E[12] \cong \bigwedge^2 E[3] \times \bigwedge^2 E[4]$, and Theorem 5.1 for $n = 3, 4$ immediately gives the analogous result for $n = 12$.

6 An isogeny of an elliptic curve and the 12-th roots of its discriminant

The following corollary gives a variant of a Coates' result [2, appendix]. The original result assumed that the characteristic of the base field is 0.

Corollary 6.1. *Let E and E' be elliptic curves over a field K of characteristic $\text{char}(K) \neq 2, 3$, and $\varphi : E \rightarrow E'$ be an isogeny over K . If $d = \deg \varphi$ is prime to 12, then we have $\Delta_E = (\Delta_{E'})^d$ in $K^\times / (K^\times)^{12}$.*

Proof. By Theorem 5.1, we have the following commutative diagram:

$$\begin{array}{ccc}
 \mu_{12} \times \mu_{12} \sqrt[12]{\Delta_E} & \longrightarrow & \mu_{12} \sqrt[12]{\Delta_E} \\
 e_{12} \times w_{12} \uparrow & & \uparrow w_{12} \\
 \bigwedge^2 E[12] \times T(E[12]) & \longrightarrow & T(E[12]) \\
 \wedge^2 \varphi \times T\varphi \downarrow & & \downarrow T\varphi \\
 \bigwedge^2 E'[12] \times T(E'[12]) & \longrightarrow & T(E'[12]) \\
 e_{12} \times w'_{12} \downarrow & & \downarrow w'_{12} \\
 \mu_{12} \times \mu_{12} \sqrt[12]{\Delta_{E'}} & \longrightarrow & \mu_{12} \sqrt[12]{\Delta_{E'}} \\
 (\cdot)^{12} \times (\cdot)^{12} \downarrow & & \downarrow (\cdot)^{12} \\
 \mu_{12} \times \mu_{12} \sqrt[12]{\Delta_{E'}^d} & \longrightarrow & \mu_{12} \sqrt[12]{\Delta_{E'}^d}.
 \end{array}$$

Here, $(\cdot)^{12}$ denotes the 12-th power map, and the horizontal maps are the action map. Also, the vertical maps in the above diagram are all bijective and G_K -equivariant. Since $e_{12} \circ \wedge^2 \varphi \circ e_{12}^{-1} = (\cdot)^d : \mu_{12} \rightarrow \mu_{12}$ (for example, see [8, III, Proposition 8.2]) and $d^2 \equiv 1 \pmod{12}$, the outside rectangle in the above diagram becomes

$$\begin{array}{ccc}
 \mu_{12} \times \mu_{12} \sqrt[12]{\Delta_E} & \longrightarrow & \mu_{12} \sqrt[12]{\Delta_E} \\
 \wr \downarrow \text{id} \times \sigma & & \wr \downarrow (\cdot)^{12} \circ w'_{12} \circ T\varphi \circ w_{12}^{-1} \\
 \mu_{12} \times \mu_{12} \sqrt[12]{\Delta_{E'}^d} & \longrightarrow & \mu_{12} \sqrt[12]{\Delta_{E'}^d}.
 \end{array}$$

This implies that $\mu_{12} \sqrt[12]{\Delta_E}$ and $\mu_{12} \sqrt[12]{\Delta_{E'}^d}$ are isomorphic as μ_{12} -torsors over K . Hence the assertion holds. \square

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